Derivatives Pricing In An Incomplete Market
Part I - Minimize The Squared Replication Error

Gary Schurman, MBE, CFA

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A market is incomplete if the number of basis assets in that market is less than the number of states of the world in that market. If a market is incomplete then the price of a redundant asset, such as a derivative, will be such that the resulting hedge will be optimal but not perfect. We will define the optimal replicating hedge as the hedge that minimizes the squared replication error.

Our Hypothetical Problem

Imagine that we have a market where there are three states of the world but only two basis assets, a stock and a risk-free bond. The table below presents the payoff matrix on the stock and bond at time t given the actual state of the world at that time and the attendant probabilities...

Table 1: Payoff Matrix

SOTW	Description	Stock	Bond	Prob
1	Good economy	20.00	1.05	0.40
2	Average economy	10.00	1.05	0.50
3	Bad economy	5.00	1.05	0.10

The market prices of the stock and bond at time zero are \$12.00 and \$1.00, respectively.

Question: We have a put option on the stock with an exercise price of \$11.00. The option can be exercised at time t. What is the value of the put option at time zero?

Constructing the Hedge Portfolio

To hedge a short position in a derivative asset the seller of the derivative can hedge their exposure by taking long and/or short positions in the basis assets that exist in that market. In a complete market the goal is to construct a hedge portfolio where the value of this portfolio at time t equals the payoff on the derivative asset at time t regardless of the actual state of the world at that time. In an incomplete market the goal is to construct a hedge portfolio that minimizes the square of the difference between the value of the portfolio at time t and the payoff on the derivative asset at time t. In a complete market we can construct the **perfect** hedge. In an incomplete market we seek to construct the **optimal** hedge. The cost to build the hedge portfolio at time zero is the value of the derivative asset at time zero.

We will define matrix $\mathbf{A}$ to be a matrix of payoffs on the basis assets at time t. The actual payoff amount is contingent on the state of the world at time t. For this matrix the matrix rows represent basis asset payoffs for each state of the world. Since we have three states of the world our matrix has three rows. Since we have two basis assets our matrix has two columns. Using the data from our hypothetical problem above the equation for the payoff matrix is...

$$\mathbf{A}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{bmatrix}=\begin{bmatrix}20.00 & 1.05\\ 10.00 & 1.05\\ 5.00 & 1.05\end{bmatrix} \tag{1}$$

Note that the transpose of matrix $\mathbf{A}$ as defined by Equation (1) above is...

$$\mathbf{A}^T=\begin{bmatrix}a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\end{bmatrix}=\begin{bmatrix}20.00 & 10.00 & 5.00\\ 1.05 & 1.05 & 1.05\end{bmatrix} \tag{2}$$

We will define vector $\vec{\mathbf{b}}$ to be the payoffs on the focus asset, which is our put option, at time t. Using the data from our hypothetical problem above the equation for the focus asset payoff vector given the three states of the world is...

$$\vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0.00 \\ 1.00 \\ 6.00 \end{bmatrix} \tag{3}$$

We will define vector $\vec{\mathbf{v}}$ to be the vector of basis asset prices at time zero. Using the data from our hypothetical problem above the equation for the basis asset price vector is...

$$\vec{\mathbf{v}} = \begin{bmatrix} \text{Stock} \\ \text{Bond} \end{bmatrix} = \begin{bmatrix} 12.00 \\ 1.00 \end{bmatrix} \tag{4}$$

We will define vector $\vec{\mathbf{w}}$ to be a vector of asset weights (i.e. the quantities of the basis assets that we long or short at time zero). This is the vector that we want to solve for. The equation for the asset weight vector is...

$$\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{5}$$

If we had a market in which the number of states of the world was less than or equal to the number of basis assets then we could price the put option such that we would have a the perfect hedge. The equation for the solution to asset weight vector $\vec{\mathbf{w}}$ given a complete market is...

$$\mathbf{A}\vec{\mathbf{w}} = \vec{\mathbf{b}} \text{ ...where... } \vec{\mathbf{w}} = \mathbf{A}^{-1}\vec{\mathbf{b}} \tag{6}$$

In the hypothetical problem above we have three states of the world but only two basis assets, which means that the two basis assets do not span $\mathbb{R}^3$. Since our basis assets do not span the market then we will not have a perfect hedge. Instead of the perfect hedge we will construct the optimal hedge, which we will define as the hedge that minimizes the sum of squared errors. To represent the equation for the imperfect hedge we will add an error term to Equation (6) above such that that equation becomes...

$$\mathbf{A}\vec{\mathbf{w}} = \vec{\mathbf{b}} + \vec{\mathbf{e}} \text{ ...where... } \vec{\mathbf{e}} = \mathbf{A}\vec{\mathbf{w}} - \vec{\mathbf{b}} \tag{7}$$

Minimizing the Squared Replication Error

Using Equations (1), (3) and (5) above the equation for the error vector in Equation (7) above is...

$$\vec{\mathbf{e}} = \begin{bmatrix} a_{11}w_1 + a_{12}w_2 - b_1 \\ a_{21}w_1 + a_{22}w_2 - b_2 \\ a_{31}w_1 + a_{32}w_2 - b_3 \end{bmatrix} \tag{8}$$

We will define the variable SRE to be the squared replication error and the variable m to be the number of states of the world. Using Equation (8) above the equation for the squared replication error is...

$$SRE = \sum_{i=1}^{m} e_i^2 = (a_{11}w_1 + a_{12}w_2 - b_1)^2 + (a_{21}w_1 + a_{22}w_2 - b_2)^2 + (a_{31}w_1 + a_{32}w_2 - b_3)^2 \tag{9}$$

The derivative of Equation (9) above with respect to w_1 is...

$$\frac{\delta\, SRE}{\delta\, w_1} = 2\, a_{11}\,(a_{11}w_1 + a_{12}w_2 - b_1) + 2\, a_{21}\,(a_{21}w_1 + a_{22}w_2 - b_2) + 2\, a_{31}\,(a_{31}w_1 + a_{32}w_2 - b_3) \tag{10}$$

The derivative of Equation (9) above with respect to w_2 is...

$$\frac{\delta\, SRE}{\delta\, w_2} = 2\, a_{12}\,(a_{11}w_1 + a_{12}w_2 - b_1) + 2\, a_{22}\,(a_{21}w_1 + a_{22}w_2 - b_2) + 2\, a_{32}\,(a_{31}w_1 + a_{32}w_2 - b_3) \tag{11}$$

To minimize the sum of squared replication errors we set the derivatives of Equations (10) and (11) above to zero and then solve those equations simultaneously. Using Appendix Equations (19) and (20) below the equations for the derivatives of the sum of squared replication errors set to zero are...

$$\begin{aligned} 0 &= (a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})w_1 + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})w_2 - (a_{11}b_1 + a_{21}b_2 + a_{31}b_3) \\ 0 &= (a_{12}a_{11} + a_{22}a_{21} + a_{32}a_{31})w_1 + (a_{12}a_{12} + a_{22}a_{22} + a_{32}a_{32})w_2 - (a_{12}b_1 + a_{22}b_2 + a_{32}b_3) \end{aligned} \tag{12}$$

Using Equations (10) and (11) above the second derivatives of Equation (9) above with respect to w_1 and w_2 are...

$$\frac{\delta^2\, SRE}{\delta\, w_1^2} = 2\Bigg(a_{11}^2 + a_{21}^2 + a_{31}^2\Bigg) > 0 \text{ ...and... } \frac{\delta^2\, SRE}{\delta\, w_2^2} = 2\Bigg(a_{12}^2 + a_{22}^2 + a_{32}^2\Bigg) > 0 \tag{13}$$

Since the second derivatives are greater than zero we know that at the point where the first derivatives are zero marks the minimum sum of squares (i.e. SRE decreases to its minimum and then increases thereafter) rather than the maximum sum of squares.

Note that we can rewrite the system of equations in Equation (12) as...

$$\begin{aligned}(a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})w_1 + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})w_2 &= a_{11}b_1 + a_{21}b_2 + a_{31}b_3\\ (a_{12}a_{11} + a_{22}a_{21} + a_{32}a_{31})w_1 + (a_{12}a_{12} + a_{22}a_{22} + a_{32}a_{32})w_2 &= a_{11}b_1 + a_{21}b_2 + a_{31}b_3\end{aligned} \tag{14}$$

We can express the system of equations in Equation (14) above in matrix:vector notation as...

$$\mathbf{A}^T \mathbf{A}\, \vec{\mathbf{w}} = \mathbf{A}^T\, \vec{\mathbf{b}} \tag{15}$$

The Solution to the Hypothetical Problem

To minimize the squared replication error we need to construct a hedge portfolio at time zero that has a $w_1 \times$ stock price dollar position in the stock and a $w_2 \times$ bond price dollar position in the bond. To do this we must first solve for vector $\vec{\mathbf{w}}$, which is the vector of asset weights. We will start by multiplying both sides of Equation (15) above by the inverse of the product of the transpose of matrix A and matrix A. This statement in equation form is...

$$\Bigg(\mathbf{A}^T \mathbf{A}\Bigg)^{-1} \mathbf{A}^T \mathbf{A}\, \vec{\mathbf{w}} = \Bigg(\mathbf{A}^T \mathbf{A}\Bigg)^{-1} \mathbf{A}^T\, \vec{\mathbf{b}} \text{ ...such that... } \vec{\mathbf{w}} = \Bigg(\mathbf{A}^T \mathbf{A}\Bigg)^{-1} \mathbf{A}^T\, \vec{\mathbf{b}} \tag{16}$$

Using Equations (1), (2), (3) and (5) above the solution to Equation (16) above is...

$$\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.36 \\ 6.19 \end{bmatrix} \tag{17}$$

Note that we can use Excel's matrix multiplication functions to perform the actual matrix multiplication. Using Equations (4) and (17) above the equation for put option price at time zero is...

$$\text{Put option price} = \vec{\mathbf{w}}^T \vec{\mathbf{v}} = \begin{bmatrix} -0.36 \\ 6.19 \end{bmatrix}^T \begin{bmatrix} 12.00 \\ 1.00 \end{bmatrix} = 1.87 \tag{18}$$

Per Equation (18) above the optimal hedge is a hedge portfolio at time zero that consists of a short position in the stock equal to $-0.36 \times \$12.00 = -\$4.32$ and a long position in the bond equal to $6.19 \times \$1.00 = \$6.19$. The net cost to set up the hedge portfolio at time zero is $\$6.19 - \$4.32 = \$1.87$. Note that the expected payout on the option at time t is $\$0.00 \times 0.40 + \$1.00 \times 0.50 + 0.10 \times \$6.00 = \$1.10$.

Appendix

A. To minimize the squared replication error (SRE) we first set Equation (10) above, which is the first derivative of the sum of squared errors with respect to w_1, equal to zero....

$$\begin{aligned} 2\, a_{11}\, (a_{11}w_1 + a_{12}w_2 - b_1) + 2\, a_{21}\, (a_{21}w_1 + a_{22}w_2 - b_2) + 2\, a_{31}\, (a_{31}w_1 + a_{32}w_2 - b_3) &= 0\\ a_{11}\, (a_{11}w_1 + a_{12}w_2 - b_1) + a_{21}\, (a_{21}w_1 + a_{22}w_2 - b_2) + a_{31}\, (a_{31}w_1 + a_{32}w_2 - b_3) &= 0\\ a_{11}a_{11}w_1 + a_{11}a_{12}w_2 - a_{11}b_1 + a_{21}a_{21}w_1 + a_{21}a_{22}w_2 - a_{21}b_2 + a_{31}a_{31}w_1 + a_{31}a_{32}w_2 - a_{31}b_3 &= 0\\ (a_{11}a_{11} + a_{21}a_{21} + a_{31}a_{31})w_1 + (a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32})w_2 - (a_{11}b_1 + a_{21}b_2 + a_{31}b_3) &= 0\end{aligned} \tag{19}$$

B. To minimize the squared replication error (SRE) we next set Equation (11) above, which is the first derivative of the sum of squared errors with respect to w_2, equal to zero....

$$\begin{aligned} 2\, a_{12}\, (a_{11}w_1 + a_{12}w_2 - b_1) + 2\, a_{22}\, (a_{21}w_1 + a_{22}w_2 - b_2) + 2\, a_{32}\, (a_{31}w_1 + a_{32}w_2 - b_3) &= 0\\ a_{12}\, (a_{11}w_1 + a_{12}w_2 - b_1) + a_{22}\, (a_{21}w_1 + a_{22}w_2 - b_2) + a_{32}\, (a_{31}w_1 + a_{32}w_2 - b_3) &= 0\\ a_{12}a_{11}w_1 + a_{21}a_{12}w_2 - a_{12}b_1 + a_{22}a_{21}w_1 + a_{22}a_{22}w_2 - a_{22}b_2 + a_{32}a_{31}w_1 + a_{32}a_{32}w_2 - a_{32}b_3 &= 0\\ (a_{12}a_{11} + a_{22}a_{21} + a_{32}a_{31})w_1 + (a_{12}a_{12} + a_{22}a_{22} + a_{32}a_{32})w_2 - (a_{12}b_1 + a_{22}b_2 + a_{32}b_3) &= 0\end{aligned} \tag{20}$$